Some Spectrum Examples

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The goal is to look at the spectrum of a few key operators coming from [GS18] and the theory of Ramanujian graphs. I would like to see explicit examples of discrete, continuous and mixed spectrum.

1 Background Theory

Consider a unital $\mathbb{C}$ or $\mathbb{R}$ algebra $\mathcal{A}$ with set of units $\mathcal{A}^*$, then an element $A \in \mathcal{A}$ has spectrum given by

$$\sigma(A) \quad := \quad \{\lambda \in \mathbb{C} : A - \lambda 1 \notin \mathcal{A}^*\}$$

We will be interested only in the following unital $\mathbb{C}$ algebras

$$\mathrm{Lin}(V, V)$$

where V is some vector space, for us one of the following $\mathbb{C}^n$, $L^2(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. The operations are pointwise addition, poinwise scaling, composition as the multiplication and the identity function as the identity. Note that we want $\mathbb{C}$ algebras and so we consider $L^2(\mathbb{R}^n)$ to be complex valued functions. In these cases we call an element of the spectrum an **eigenvalue** if in adition there is an element $X \in V$ such that $AX = \lambda X$.

Any old linear operator can have an arbitrary spectrum (is that actually true? I doubt any subset of $\mathbb{C}$ is obtainable?) and so we will usually place some conditions on the operator such as bounded or self adjoint, for these properties to make sense we need an innerproduct or a norm, and for them to be useful we need the vector space to be *complete* with respect to the norm. Thankfully all the spaces we are considering here are *Hilbert spaces*, that is they are complete innerproduct spaces. On $\mathbb{C}^n$ we have the normal dot product and on L^2 we have convolution

$$\langle f, g \rangle \;\; := \;\; \int f(x)\overline{g(x)}\mathrm{d}x$$

1.1 Bounded Self Adjoint Operators

Recall that a linear map between two normed linear spaces is called bounded if there is some constant $C \in \mathbb{R}^+$ such that for every entry $v \in V$

$$\|Tv\| \leq C\|v\|$$

So the operator increases the norm less than linearly. The idea is that it sends bounded sets to bounded sets. If one takes the infimum over all such C then one gets the operator norm of T.

Theorem. *A linear operator is bounded iff it is continuous.*

A map A between two inner product spaces is called self adjoint if

$$\langle Af, g \rangle = \langle f, Ag \rangle$$

A bounded linear transfomation is compact if it sends bounded sets to precompact sets (sets with a compact closure). Compact operators are in particular bounded.

Lets consider the finite dimensional case. A linear map $\mathbb{C}^n \to \mathbb{C}^n$, which can also be considered as an element of $L^2[n^2]$ which is just functions on a finite set of points (consider the matrix entries), is given by a matrix i.e. the following is an isomorphism

$$\mathrm{Lin}(\mathbb{C}^n, \mathbb{C}^n) \to \mathrm{Mat}_{n\times n}(\mathbb{C})$$

$$f \mapsto [f(e_1) \;\cdots\; f(e_n)]$$

It is clear that any linear transformation between finite dimensional vector spaces is continuous and hence bounded. We can see this more concretely by looking at $A = \begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix}^T$, $v = (v_1, ..., v_n)^T$

$$|Av|^2 = |(A_1{\cdot}v, \cdots, A_n{\cdot}v)|^2 = \sum_i |A_i{\cdot}v|^2 \leq \sum_i |\sup_j(a_{ij})(v_1{+}\cdots{+}v_n)|^2 \leq \sup_{i,j}(a_{ij})^2 \sum_i |v|^2 \leq \sup_{i,j}(a_{ij})^2 n|v|^2$$

So the matrix is bounded in the operator sense.

For a matrix to be self adjoint is non-trivial, first the innerproduct on $\mathbb{C}^n$ is given by the usual dot product, with taking the conjugate on the second variable

$$\langle f, g \rangle = f \cdot \bar{g}$$

therefore a matrix is self adjoint iff it is Hermitian, that is equal to ites own conjugate transpose.

1.2 Spectral Theorem

Theorem. *A bounded operator between one of the spaces above has the propertie that its spectrum is bounded and closed.*

Theorem. *If in adition the opperator is self adjoint then*

- *The spectrum is real*
- *Eigenspaces are orthogonal*
- *There is no residual spectrum*

Theorem. *If in addition the operator is compact then then there exists a countable orthonormal basis of V given by eigenfunctions of the operator.*

Remark. Im pretty sure this is all fine over $\mathbb{R}$ as well, it is in finite dimensions anyway.

Remark. There is also a generalisation to just self adjoint operators although it is more complex and I dont want to go into it.

1.3 Decomposition of the Spectrum

In the context of physics I tihnk continous and discrete are simply the isolated points in $\mathbb{C}$ and the parts of the spectrum that contain an interval. Mathematically there is a third type that of points of accumulation, but I guess that they would probably just ignore that?

Because I always forget in the context of representation theory we have the discrete spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, considered as a representation of $G(\mathbb{Q})\backslash G(\mathbb{A})$ acting by right translations (remember that at least in the finite dimensional case this regular representation contains all others), which is the part of the representation that decomposes as a direct sum. Recall the last chapter of Folland where we know that this representation decomposes into a direct integral. Therefore we can ask which part is a direct sum and which part is strictly a direct integral. This continuous and discrete part might be related to the spectrum of the *measure* over which the integral giving the decomposition of the representation is taken, although Im not sure. All of that is to say that this is not relevant, as there is not any operator in sight.

The other decomposition of the spectrum is the one actually given in functional analysis classes

$$\sigma(A) = \sigma_p(A) \sqcup \sigma_{cont}(A) \sqcup \sigma_{res}(A)$$

where we have the point spectrum, continuous spectrum and residual spectrum respectively. The point spectrum is by definition simply the eigenvalues of the operator, and we will restrict to the case of bounded self adjoint operators, which have no residual spectrum and so we can simply define the continuous spectrum as the compliment of the Eigenvalues. It would normally be defined as the λ such that $A - \lambda 1$ is injective with a dense image, while the residual spectrum is when it is injective but does not have a dense image. Eigenvalues are therefore when it is simply not injective.

2 Finite Dimensional Examples

In finite dimensions we know that all the operators are clearly compact (bounded and compact are the same in finite dimensions). Moreover there can be no continuous spectrum either as an injective linear map between finite dimensional vector spaces is always surjective. So we know for Hermitian matrices there is an orthonormal basis of $\mathbb{C}^n$ given by eigenvectors and so for dimensional reasons there is n eigenvalues. In essence Hermitian matricies are diagonalisable.

2.1 A Random Hermitian Matrix

2×2 is enough to see. Lets just make something Hermitian

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Note that the diagonal must clearly be real. To diagonalise we just need to find the eigenvalues of this matrix by solving

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - (-i^2) = \lambda(\lambda - 2) = 0$$

which has solutions $\lambda = 0, 2$. And then find their eigenvectors:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = 0 \begin{pmatrix} x \\ y \end{pmatrix}$$

solving then $x + iy = 0$ and $-ix + y = 0$ gives $y = ix$. And for the other eigenvalue

$$A \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

which gives $x + iy = 2x$ and $-ix + y = 2y$, i.e. $x = iy$. Thus we have the eigenspaces

$$0 \to \begin{pmatrix} x \\ ix \end{pmatrix}, 2 \to \begin{pmatrix} ix \\ x \end{pmatrix}$$

And so

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

2.2 Symmetric But Not Hermitian Fails

Consider the example

$$\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

Then its characteristic polynomial is given by just by λ^2 and hence has a single eigenvalue 0. The eigenvectors will have to satisfy $ix = y$ and so there is only one eigenvector and we cannot form the matrix to diagonalise the operator. This is the existence of Jordan blocks.

3 Infinite Dimensional Examples

Here are the examples in [GS18]. I will set $\hbar$ and the mass to be 1, or whatever they need to be such that all constants in the formula are 1. Lets consider only things in one dimension. Recall that

$$\hat{H} = \frac{\partial^2}{\partial x^2} + V(x)$$

Note that this the $V(x)$ part acts on a function by multiplication. This is the Hamiltonian, the energy operator. There are several V specified, for instance the infinite square well (which should be the same as considering $V = 0$ and $\hat{H}$ as an operator $L^2[0,1] \to L^2[0,1]$). The Harmonic oscillator $V(x) = x^2$.

Two important multidimensional examples are the angular momentum and Laplacian. For angular momentum the operators are

$$L_z = xp_y - yp_x$$

or similarly for L_x, L_y and we also define $L^2 = L_x^2 + L_y^2 + L_z^2$. The laplacian is simpler

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

This has other generalisations to manifolds and higher dimensions; understanding its generalisationt to manifolds will lead us to psuedo-differential operators and the Aitiyah-Singer index theorems.

Note that it is clear that if we multiply a function by a scalar then it preserves the properties of compactness or boundedness, however this is not the case for *self adjoint*, indeed even in the finite dimensional case we can see that if we multiplied our Hermitian matrix above by i it is no longer Hermitian, or multiplication by ix as an operator. Multiplication by real scalars will preserve this though. In this section we will consider operators on the vector spaces $L^2(\mathbb{R}^n)$ or $H^n(\mathbb{R})$ (Sobolev space, that is functions that are n differentiable and whose derivatives are also integrable), that is smooth and integrable functions.

$i\frac{\partial}{\partial x}$ is self adjoint for reasons beyond me. Note that the composition of self adjoint operators is self adjoint if the operators commute. Therefore the Laplacian on $\mathbb{R}$ is self adjoint. By linearity of the innerproduct the sum of self adjoint operators is also self adjoint so the Laplacian in mlutiple dimensions is also self adjoint.

None of these operators are bounded on $\mathbb{R}$ (so cannot be compact either). Multiplication by x is bounded on $[0,1]$ (or any other set of finite measure). The derivative is basically never bounded. Regardless they have a spectrum that we can find.

Graphs for each.

Remark. Examples I can think of on an infinite domain $L^2(\mathbb{R})$ is basically scaling by a real constant $f \mapsto mf$. After talking to an analyst they told me that all compact operators are going to be limits of finite rank operators. The finite rank operators will be given by projections onto finite dimensional subspaces. So in a sense, at least in this case, the only other non-trivial compact operator will be projection onto a linear combination of subspaces (or limits thereof). Im not sure to what extent this is precisely true but at least in spirit.

Remark. Adding a constant *is not linear*! It doesnt map the identity to the identity.

3.1 Scaling

Consider the operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by $f \mapsto af$ for $a \in \mathbb{R}$, then this is clearly a compact self adjoint operator. In fact it is given by $a1$ and hence $a1 - \lambda 1 = (a - \lambda)1$ which is clearly invertible (with inverse $\frac{1}{a-\lambda}1$) for any $\lambda \in \mathbb{C} \setminus \{a\}$, so its spectrum

$$\sigma(a1) = \{a\}$$

It is clear that any non-zero function is an eigenvalue and hence we have only a pure point spectrum. Because the whole space is the eigen space the other results of the theorems are also immediate (it has an orthonormal basis).

This is similar to the finite dimensional case of $a1$ which is already diagonal and hence clearly has eigenvalues a. Moreover if we solve $a1X = aX$ it is all X.

3.2 Multiplication by x

This is a self adjoint operator on $L^2(X)$ for $X \subseteq \mathbb{R}$. First let us look at $X = [0,1]$. From the physical perspective this is the position operator for a particle in an infinite well. We claim that it has no point spectrum (no eigenfunctions) and that its spectrum is $[0,1]$. An eigenfunction would be a function on $[0,1]$ such that

$$xf(x) = \lambda f(x)$$

if we assume that $f(x) \neq 0$ for a particular x then we get that

$$x = f(x)/f(x) = 1$$

thus the only function that can satisfy this must be zero for all $x \neq 1$, or up to L^2 equivalence this is only the zero function. So any spectrum it has must be continuous (it is bounded and self adjoint so has no residual spectrum). Calling this operator T we now want to look at when

$$T - \lambda 1$$

is invertible. If $\lambda \notin [0,1]$ then we have the operator

$$f \mapsto \frac{1}{x - \lambda} f$$

where

$$(T - \lambda 1)(\frac{1}{x - \lambda} f) = \frac{x}{x - \lambda} f - \frac{\lambda}{x - \lambda} f = \frac{x - \lambda}{x - \lambda} f = f$$

This makes sense because $\frac{1}{x-\lambda}$ is defined **and integrable** for all $x \in [0,1]$. Conversely we can see that if $\lambda \in [0,1]$ then the operator $T - \lambda 1$

$$(T - \lambda 1)f(x) = xf(x) - \lambda f(x) = (x - \lambda)f(x)$$

can have no inverse, as for instance there it is not surjective: no f *integrable* exists such that

$$(x - \lambda)f(x) = 1$$

It will have a pole in the domain that is not integrable.

It is clear from this discussion that the same is true for $L^2(\mathbb{R})$ as well, its spectrum is $\mathbb{R}$ and it is pure continuous spectrum.

3.3 Partial Derivative

Here we want to find the spectrum of $i\frac{\partial}{\partial x}$ which is an unbounded self adjoint operator on say $H^n(X)$ (Sobolev space) for $X \subseteq \mathbb{R}$. This is the momentum operator.

Lets treat $X = [0,1]$ first. Lets look for point spectrum first, that is solutions to

$$i\frac{\partial}{\partial x}f = \lambda f \iff \frac{\partial}{\partial x}f = -i\lambda f$$

This is just a linear seperable first order ODE, with the well known solution

$$\exp(-i\lambda x).$$

This is well defined for any $\lambda \neq 0$ (for which the solutions become just the constant functions). Both the constant functions and e are integrable over $[0,1]$. Therefore we have only point spectrum given by the whole real line.

If we now look at $L^2(\mathbb{R})$ then we have the same differential equation however the solutions are no longer $L^2(\mathbb{R})$ hence there is no point spectrum any more. We claim however that the continuous spectrum is now all of $\mathbb{R}$. Namely that

$$i\frac{\partial}{\partial x} - \lambda 1$$

is never invertible. The details are in the link above, however the idea is that you can use a cutoff function on the eigenvectors given above in the finite domain and then use the fact that the operator has an inverse to bound the norm of this function by $1/n$ for every n and therefore see that the function must be zero.

Remark. So changing the domain from $H^1[0,1]$ to $H^1(\mathbb{R})$ just changes all the point spectrum into the continuous spectrum, but they have the same spectrum!

Remark. Note also that in the $[0,1]$ case we get the eigenfunctions of $e^{i\theta}$, the fact that this is an orthonormal basis of L^2 is essentially the Fourier transform / Dirichlets theorem.

3.4 Laplacian

We want the spectrum of $\Delta := -\nabla^2$ as an operator on $H^k(X)$. Again this is self adjoint but unbounded. In this case there is very different behaviour depending on the domain and depending *on the boundary conditions*. The infinite square well has a Hamiltonian given by the Laplacian on $L^2[0,1]$ (say) with boundary conditions called Dirichlet boundary conditions and they force the spectrum to be discrete, essentially an operator on $L^2[0,1] \cap$zero on the boundary. The free particle has a Hamiltonian given by the Laplaican on $L^2(\mathbb{R}^n)$. The last interesting case we will consider is the Hamiltonian for *the finite* square well, which has a mixed spectrum.

There is a subtlety with boundary conditions and self adjointness aparently; Source is ChatGPT. If we dont fix boundary conditions on the finite interval then the Laplacian is not self adjoint, it admits a non-unique self adjoint extension that somehow corresponds to enforcing boundary conditions. This doesnt appear in the unbounded setting because for the functions to be L^2 they must be zero "on the boundary" of $\mathbb{R}$ anyway.

3.4.1 $\mathbb{R}^n$

There is a nice discussion here. The idea is that if we denote the Fourier transform $\mathcal{F}$ then

$$\mathcal{F}\Delta\mathcal{F} = M_{|x|^2}$$

where M_h is the opperator given by multiplying by h (a function). By the Plancherel theorem the Fourier transform is a unitary map (preserves norms) and conjugating an operator by a unitary transformation will not change the spectrum, therefore

$$\sigma(\Delta) = \sigma(M_{|x|^2})$$

Now there is a general theory (apparently in Reed and Simon) for finding the spectrum of multiplication operators, it is given by the essential range of the multiplying function (the essential range is a technical condition on the image of a measurable function, taking values in the codomain such that every neighbourhood of them has a positive measure in the preimage). It is clear that the (essential) range of $|x|^2$ is the non-negative numbers $[0, \infty)$. Moreover this is pure continuous spectrum.

3.4.2 $[0, 1]$ with boundary conditions

The two types of boundary conditions are Dirichlet (requiring the functions to be zero on the boundary) and Von Neuman (requiring the derivative of the functions to be zero on the boundary). Both produce discrete spectrum, we will consider the Dirichlet boundary conditions.

First we solve for the point spectrum $-f'' = \lambda f$. This is a second order linear ODE that can be solved by people who know how to do such things. For positive λ the general solution is

$$a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$$

and the boundary conditions give

$$a\cos(0) + b\sin(0) = a = 0$$

which means the solution is of the form

$$b\sin(\sqrt{\lambda}) = 0 \implies \sin\sqrt{\lambda} = 0$$

for a non-trivial eigenvalue. This implies that $\lambda = (n\pi)^2, n \in \mathbb{N}$ (or $\mathbb{Z}$ because we are taking the square).

If λ is not one of these values then an explicit inverse to the function $\Delta - \lambda 1$ can be constructed using Greens functions. There are other arguments for the spectrum being discrete in this case using resolvents.

3.4.3 The finite well

Following these slides. We define the potential function to be

$$V(x) \quad := \quad \begin{cases} 0, & |x| > a \\ V_0, & \text{else} \end{cases}$$

then the time independent Schrodinger equation is

$$(\Delta - V)\psi = 0$$

Put a picture of the potential. The Hamiltonian is therefore $\Delta - V$ and the spectrum correpsonds to the possible energy values. First we solve for the point spectrum

$$(\Delta - V)f = \lambda f$$

There is a continuous spectrum $[0, \infty)$ and a finite number of negative point spectrum. An intuition for this can be gained from the functional calculus. We are applying a function that subtracts a finite number *in a bounded region* and so the spectrum should also shift down but only for a finite number of points. The exact proof is beyond me at this stage.

References

[GS18] David J. Griffiths and Darrell F. Schroeter. *Introduction to Quantum Mechanics.* Cambridge University Press, 3 edition, August 2018.